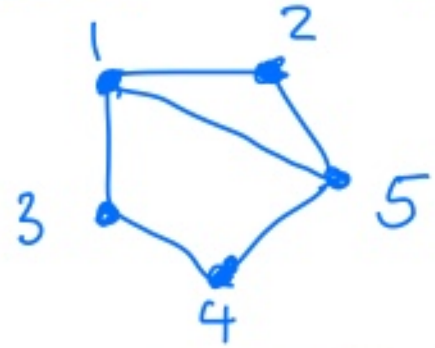
Question: In the graph below ① Find the adjacency matrix.

② How many paths go from vertex 2 to vertex 3 that have length 3?



Answers ①

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

② $2 \to 5 \to 4 \to 3$ (2 ways)

$2 \to 5 \to 1 \to 3$

Given a graph G, a path is a sequence of edges of the form

(undirected) $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\} \cdots \{v_n, v_{n+1}\}$.

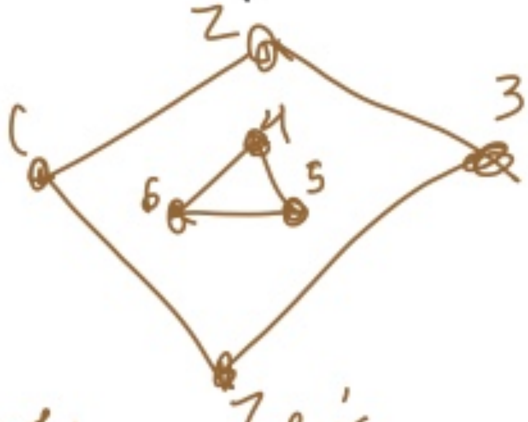
(directed) $(v_1, v_2), (v_2, v_3), \ldots \quad (v_n, v_{n+1})$

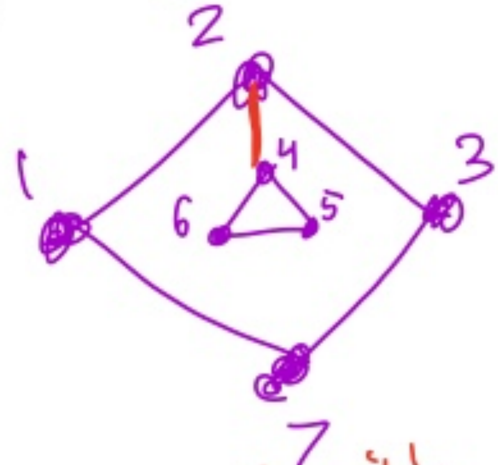
These are called paths of length n, because they include n edges.

Note: We could repeat the same edge more than once, and also vertices can be used multiple times.

A closed path is a path that starts and ends at the same vertex. (Another name for closed path is circuit.) A simple path is a path with no repeated edges.

A graph is called connected if we can find a path between any two vertices.





This graph is disconnected (ie. not connected), because for instance there is no path from vertex 3 to vertex 5.

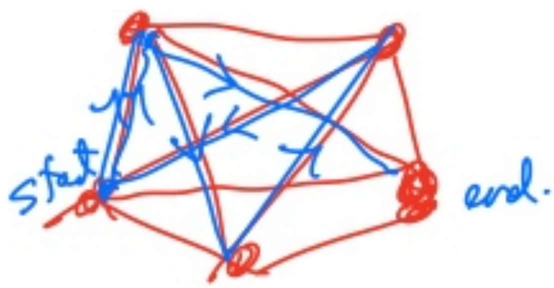
Now it's connected

Theorem: If a graph G is connected, then there exists a simple path between any two vertices.

In fact, we can connect any two vertices with a simple path where no vertices are repeated.

Proof:

Idea



Sketch of the proof: Suppose there is a path between vertices a & b in G, given by $\{\underset{v_1}{a}, v_2\}, \{v_2, v_3\}, \ldots \{v_n, \underset{v_{n+1}}{b}\}$

If $v_j = v_k$ for some $j < k$, then this path also works.

$$\{a, v_2\}, \ldots, \{v_{j-1}, \overset{\text{same}}{\overbrace{v_j\}, \{v_k}}, v_{k+1}\}, \ldots \{v_n, b\}$$

Now we have 1 less repetition of vertices. Keep shortening the path until there are no remaining repeated vertices. ▨

Corollary: If G has N vertices — then if G is connected, then there exists a simple path between any two vertices of length $\leq N$.

Measures of how connected a graph is:

A vertex cut of a graph G is a set of vertices, that, when removed from G (including the edges attached to those vertices), the resulting graph is disconnected. An edgecut of a graph G a set of edges such that, when they are removed, the graph is disconnected.

A couple of invariants of Graphs related to connectivity:

$\kappa(G)$ = minimum # of vertices in a vertex cut of a graph

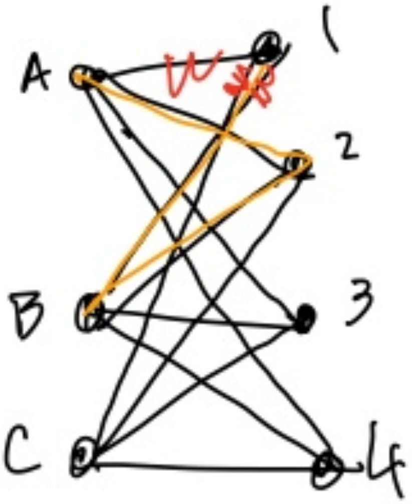
$\kappa(K_n) := n-1$

$\lambda(G)$ = minimum # of edges in an edge cut of a graph.

Thm. - For a given graph,

$$\kappa(G) \le \lambda(G) \le \begin{pmatrix}\text{minimum deg}\\ \text{of a vertex}\\ \text{on the graph}\end{pmatrix}.$$

Example $K_{3,4}$



edgecut - delete $\{A,1\}, \{B,1\}, \{C,1\}$

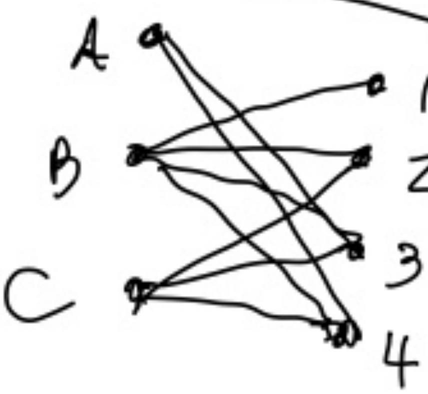
1 disconnected from the rest

in fact $\lambda(G)=3$

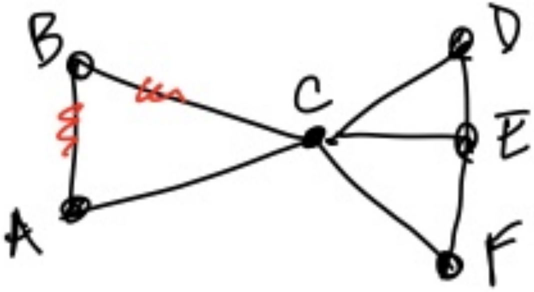
vertex cut delete A, B, C

$\kappa(G)=3$

mindeg $=3$



edge cut $=\{B,1\}$.



vertex cut : remove C.

$K(G)=1$

$\lambda(G)=2$

edge cut: $\{A,B\}$, $\{B,C\}$.

How is this related to Adjacency matrices?

<u>Aside</u>: Matrix multiplication

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1(-1)+2(4) & 1(2)+(2)5 \\ 0\cdot(-1)+1\cdot 4 & 0\cdot 2+1\cdot 5 \end{pmatrix}$$

row column

2nd row 1st column

row · column

$$= \begin{pmatrix} 7 & 12 \\ 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} \ \end{pmatrix}\begin{pmatrix} \ \end{pmatrix} = \begin{pmatrix} \ \end{pmatrix}$$

$n \times k$ (row, column) $k \times r$ (row, column) $n \times r$

Quiz

1. What is your favorite ice cream flavor?

2. Find the adjacency matrix of this graph